17. Standard Inner Product

Wei-Shi Zheng, 
wszheng@ieee.org, 2011

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1 What Do You Learn from This Note

Up to this point, the theory we have established can be applied to any vector space in general. However, vector spaces over \( \mathbb{R} \) have geometric structure on their own. You probably have already known concepts such as lengths, angles and distances of vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). Now we shall see how these geometric concepts can be generalized to vector spaces over \( \mathbb{R} \), particularly \( \mathbb{R}^n \).

Basic Concept: inner product (内积), length (长度), unit vector (单位向量), distance (距离), orthogonality (正交), orthogonal complements (正交补)

2 Inner Product & Length

2.1 Basic Concept

Definition 1 (inner product or dot product, 内积). Let \( \vec{u}, \vec{v} \in \mathbb{R}^n \). The standard inner product or dot product \( \vec{u} \cdot \vec{v} \) of \( \vec{u} \) and \( \vec{v} \) is defined to be the real number \( \vec{u}^T \vec{v} \).

Definition 2 (norm or length, 模或长度). Let \( \vec{v} \in \mathbb{R}^n \). The norm or length \( \| \vec{v} \| \) of \( \vec{v} \) is defined to be

\[
\| \vec{v} \| = \sqrt{\vec{v} \cdot \vec{v}},
\]
which is a non-negative real number.

**Definition 3** (UNIT VECTOR, 单位向量). *is a vector whose norm is 1. If we are given any non-zero vector \( \vec{v} \), then \( \frac{\vec{v}}{||\vec{v}||} \) is a unit vector, which is called the unit vector in the same direction as \( \vec{v} \). The process of creating \( \frac{\vec{v}}{||\vec{v}||} \) is called normalisation.*

**Definition 4** (ANGLE). Let \( \vec{u}, \vec{v} \in \mathbb{R}^n - \{\vec{0}\} \). The angle \( \theta \) between \( \vec{u} \) and \( \vec{v} \) is defined to be

\[
\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||},
\]

which is a number in the interval \( [0, \pi] \).

**Definition 5** (DISTANCE). Let \( \vec{u}, \vec{v} \in \mathbb{R}^n \}. The distance \( d(\vec{u}, \vec{v}) \) between \( \vec{u} \) and \( \vec{v} \) is defined to be

\[
d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}||.
\]

**Definition 6** (ORTHOGONALITY). Two vectors \( \vec{u} \) and \( \vec{v} \) in \( \mathbb{R}^n \) are orthogonal if \( \vec{u} \cdot \vec{v} = 0 \).

### 2.2 Some Properties

**Theorem 7.** Let \( \vec{u}, \vec{v} \in \mathbb{R}^n \) and \( c \in \mathbb{R} \). Then

1. \( \vec{u} \cdot \vec{u} \geq 0 \) and \( \vec{u} \cdot \vec{u} = 0 \) iff \( \vec{u} = \vec{0} \);
2. \( \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \);
3. \( \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \);
4. \( \vec{u} \cdot c \vec{v} = c(\vec{u} \cdot \vec{v}) \).

(So we say \( \mathbb{R}^n \) equipped with dot product is an inner product space.)

**Proof.** Straightforward. \( \square \)

**Remark:** 1) 3. and 4. of Theorem 2 indicate that dot product is linear on the second operand, and 2. ensures that dot product is also linear on the first operand.
2) We have $\|v\| = 0$ iff $v = \vec{0}$.
3) For any scalar $c$, we have $\|cv\| = \sqrt{c^2} = \sqrt{c^2\|v\|^2} = |c|\|v\|$.

Example: Textbook P.377.

**Theorem 8 (the Cauchy–Schwarz inequality).** Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then

$$(\vec{u} \cdot \vec{v})^2 \leq (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}), \text{ or equivalently, } |\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|.$$ 

The equality holds if and only if $\vec{u}$ and $\vec{v}$ are linearly dependent.

**Proof. Step 1.** Suppose first that $\vec{u}$ and $\vec{v}$ are linearly independent. Then neither $\vec{u}$ nor $\vec{v}$ is $\vec{0}$. Let $t$ be any scalar. Then $\vec{u} - tv \neq \vec{0}$, so we have

$$0 < (\vec{u} - tv) \cdot (\vec{u} - tv) \quad \text{[By 1. of Theorem 2.]}$$

$$= (\vec{u} \cdot \vec{u}) - t(\vec{u} \cdot \vec{v}) - t(\vec{v} \cdot \vec{u}) + t^2(\vec{v} \cdot \vec{v}) \quad \text{[By linearity.]}$$

$$= (\vec{u} \cdot \vec{u}) - 2t(\vec{u} \cdot \vec{v}) + t^2(\vec{v} \cdot \vec{v}) \quad \text{[By symmetry.]}$$

By the arbitrariness of $t$, we may take $t = \frac{(\vec{u} \cdot \vec{v})}{(\vec{v} \cdot \vec{v})}$ and substitute it into the above inequality to obtain

$$0 < (\vec{u} \cdot \vec{u}) - 2\frac{(\vec{u} \cdot \vec{v})^2}{(\vec{v} \cdot \vec{v})}(\vec{v} \cdot \vec{v}) + (\vec{u} \cdot \vec{v})^2$$

$$= (\vec{u} \cdot \vec{u}) - 2\frac{(\vec{u} \cdot \vec{v})^2}{(\vec{v} \cdot \vec{v})} + (\vec{u} \cdot \vec{v})^2$$

$$= (\vec{u} \cdot \vec{u}) - (\vec{u} \cdot \vec{v})^2.$$ 

Multiply $(\vec{v} \cdot \vec{v})$ on both side, we get

$$0 < (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2.$$ 

So $(\vec{u} \cdot \vec{v})^2 < (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v})$.

**Step 2.** If $\vec{u}$ and $\vec{v}$ are linearly dependent then either $\vec{u} = \lambda \vec{v}$ or $\vec{v} = \lambda \vec{u}$. For both cases, it is obvious that $(\vec{u} \cdot \vec{v})^2 = (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v})$. 

By the Cauchy–Schwarz inequality, for any $\vec{u}, \vec{v} \in \mathbb{R}^n - \{\vec{0}\}$ we have

$$-1 \leq \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \leq 1.$$ 

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Remarks:
1. By definition, we have $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$.
2. By Cauchy–Schwarz inequality, $\theta = 0$ or $\pi$ iff $\vec{u}$ and $\vec{v}$ are linearly dependent. It is not hard to see that $\theta = 0$ if $\vec{u}$ and $\vec{v}$ are in the same direction and $\theta = \pi$ if $\vec{u}$ and $\vec{v}$ are in the opposite direction.

**Theorem 9.** Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then
1. $|\vec{u}| \geq 0$ and $|\vec{u}| = 0$ iff $\vec{u} = \vec{0}$;
2. $|c\vec{u}| = |c||\vec{u}|$;
3. $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$ and the equality holds iff $\vec{u}$ and $\vec{v}$ are in the same direction.
(So we say $\mathbb{R}^n$ equipped with $\| \cdot \|$ is a normed space.)

**Proof.** 1. and 2. have been shown already. For 3. we have

$$|\vec{u} + \vec{v}|^2 = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} + 2(\vec{u} \cdot \vec{v}) \leq |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}||\vec{v}| \quad \text{[by Cauchy–Schwarz inequality.]}
$$

$$= (|\vec{u}| + |\vec{v}|)^2
$$

So $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$ and the equality holds iff $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|$ iff $\vec{u}$ and $\vec{v}$ are in the same direction. \(\square\)

Example: Textbook P.378.

**Theorem 10.** Let $\vec{u}, \vec{v}$ and $\vec{w} \in \mathbb{R}^n$. Then
1. $d(\vec{u}, \vec{v}) \geq 0$ and $d(\vec{u}, \vec{v}) = 0$ iff $\vec{u} = \vec{v}$;
2. $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$;
3. **Triangle inequality:** $d(\vec{u}, \vec{w}) \leq d(\vec{u}, \vec{v}) + d(\vec{v}, \vec{w})$.
(So we say $\mathbb{R}^n$ equipped with $d(\cdot, \cdot)$ is a metric space.)

**Proof.** 1. and 2. are obvious. For 3., we have

$$d(\vec{u}, \vec{w}) = \|\vec{u} - \vec{w}\| = \|(\vec{u} - \vec{v}) + (\vec{v} - \vec{w})\| \leq \|\vec{u} - \vec{v}\| + \|\vec{v} - \vec{w}\| = d(\vec{u}, \vec{v}) + d(\vec{v}, \vec{w}).$$

\(\square\)

3 Orthogonality: More

**Definition 11** (orthogonality,正交). Vectors $\vec{u}$ and $\vec{v}$ are said to be **orthogonal**, written as $\vec{u} \perp \vec{v}$, iff $\vec{u} \cdot \vec{v} = 0$. If neither $\vec{u}$ nor $\vec{v}$ is $\vec{0}$, then $\vec{u} \perp \vec{v}$ is equivalent to $\theta = \frac{\pi}{2}$. 

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Theorem 12. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$.
1. the Cosine Theorem: If neither $\vec{u}$ nor $\vec{v}$ is $\vec{0}$, then
   $$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos \theta.$$ 
2. the Pythagorean Theorem: If $\vec{u} \perp \vec{v}$, then
   $$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Proof. 1.
$$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2(\vec{u} \cdot \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos \theta.$$ 
2.
$$\|\vec{u} + \vec{v}\|^2 = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} + 2(\vec{u} \cdot \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Definition 13 (orthogonal complements, 正交补). If a vector $\vec{z}$ is orthogonal to every vector in a subspace $W$ of $\mathbb{R}^n$, then $\vec{z}$ is said to be orthogonal to $W$. The set of all vector $\vec{z}$ that are orthogonal to $W$ is called the orthogonal complement of $W$ and is denoted by $W^\perp$.

Theorem 14. Let $W$ be a subspace of $\mathbb{R}^n$. Then $W^\perp$ is a subspace.

Proof. Exercise.

Theorem 15. Let $A$ be an $m \times n$ matrix. Then $(\text{Row}A)^\perp = \text{Nul}A$ and $(\text{Col}A)^\perp = \text{Nul}A^T$