16. Eigenvalues and Eigenvectors

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1 What Do You Learn from This Note

In this lecture note, we are considering a very special matrix equation for
a given square matrix $A$:

$$A\vec{x} = \lambda\vec{x}.$$  \hspace{1cm} (1)

This equation is very important for developing optimisation algorithm for
many engineering problems(大家将来遇到很多的科学计算优化问题都可以
归结为解这个特征向量方程的问题).

Basic Concept: Eigenvalue(特征值), Eigenvector(特征向量), Characteristic
equation(特征方程)

2 Eigenvalue & Eigenvector

Definition 1 (EIGENVALUE(特征值) AND EIGENVECTOR(特征向量)). Let
$A$ be a $n \times n$ square matrix. An eigenvalue $\lambda$ of $A$ is a scalar such that
$A\vec{x} = \lambda\vec{x}$ for some non–zero vector $\vec{x}$, which is called an eigenvector of $A$
corresponding to $\lambda$ (or a $\lambda$-eigenvector).

Example: For any $\vec{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we have $(\lambda I_n)\vec{x} = \lambda\vec{x}$. So $\lambda$ is an
eigenvalue of $\lambda I_n$ and any non–zero vector is a $\lambda$–eigenvector.

Example: Let $A \in \mathbb{R}^n$ be non–invertible, which is equivalent to Nul$(A) - $
$\{0\} \neq \emptyset$. Then we have $A\vec{x} = \vec{0} = 0\vec{x}$ for any $\vec{x} \in \text{Nul}(A) - \{0\}$. So $0$ is an
eigenvalue of $A$ and any non-zero vector in $\text{Nul}(A) - \{\vec{0}\}$ is a 0-eigenvector.

**Warning:** By definition, eigenvalue can take zero scalar whereas eigenvector is restricted to non-zero vectors.(特征值可以是0，但特征向量不能是零向量)

Examples: Textbook P.303.

**Theorem 2.** If $\vec{v}_1, \ldots, \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix $A$, the set $\vec{v}_1, \ldots, \vec{v}_r$ is linearly independent.

**Proof.** We conduct the proof by contradiction (我们用反正法) and induction method.

1. STEP 1: We first prove for two eigenvectors $\vec{v}_{i_1}$ and $\vec{v}_{i_2}$ that correspond to different eigenvalues $\lambda_{i_1}, \lambda_{i_2}$, $\vec{v}_{i_1}$ and $\vec{v}_{i_2}$ must be linearly independent. If not, then we have $\vec{v}_{i_1} = c\vec{v}_{i_2}$ for some $c \neq 0$. Also, $A\vec{v}_{i_1} = cA\vec{v}_{i_2}$, hence $\lambda_{i_1}\vec{v}_{i_1} = c\lambda_{i_2}\vec{v}_{i_2}$. Combing with $\lambda_{i_1}\vec{v}_{i_1} = c\lambda_{i_2}\vec{v}_{i_2}$, so $(\lambda_{i_1} - \lambda_{i_2})\vec{v}_{i_2} = \vec{0}$. Since $\lambda_{i_1} \neq \lambda_{i_2}$, hence $\vec{v}_{i_2} = \vec{0}$, which is impossible, as an eigenvector must be non-zero.

2. STEP 2: for any $r-1$ eigenvector, $\vec{v}_{i_1}, \ldots, \vec{v}_{i_{r-1}}$ corresponding to $r-1$ distinct eigenvalues are linearly independent.

3. SETP 3: for given $r$ eigenvectors, If these $r$ vectors are linearly dependent, then there exists a vector $\vec{v}_j$ such that $\vec{v}_j$ is a linear combination of the other eigenvectors, that is there are a series of weights $c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_r$ some of which are not zero having

$$\vec{v}_j = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_r\vec{v}_r. \quad (2)$$

Also by multiplying $A$ on both side, we have

$$\lambda_i\vec{v}_i = c_1\lambda_1\vec{v}_1 + \cdots + c_{i-1}\lambda_{i-1}\vec{v}_{i-1} + c_{i+1}\lambda_{i+1}\vec{v}_{i+1} + \cdots + c_r\lambda_r\vec{v}_r. \quad (3)$$

In addition, by multiplying $\lambda_i$ on both sides of Eq. (2), then we have

$$\lambda_i\vec{v}_i = c_1\lambda_i\vec{v}_1 + \cdots + c_{i-1}\lambda_i\vec{v}_{i-1} + c_{i+1}\lambda_i\vec{v}_{i+1} + \cdots + c_r\lambda_i\vec{v}_r. \quad (4)$$

Combining Eqs. 3 and 5, then we have

$$\vec{0} = c_1(\lambda_i - \lambda_1)\vec{v}_1 + \cdots + c_{i-1}(\lambda_i - \lambda_{i-1})\vec{v}_{i-1} + c_{i+1}(\lambda_i - \lambda_{i+1})\vec{v}_{i+1} + \cdots + c_r(\lambda_i - \lambda_r)\vec{v}_r. \quad (5)$$

2
As the \( r - 1 \) eigenvectors \( \vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_r \) are linearly independent, and not all the weights \( c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_r \) are zero, then there must be some weight \( c_k \neq 0 \) so that \( \lambda_i - \lambda_k \). This contradicts to the statement that all the eigenvectors are distinct. Hence all the \( r \) eigenvectors that correspond to \( r \) different eigenvalues are linearly independent.

\( \square \)

3 Computation of Eigenvalues and Eigenvectors

Lemma 3. Let \( A \) be a \( n \times n \) square matrix. Then for any scalar \( \lambda \), we have

\[
\{ \vec{x} | A \vec{x} = \lambda \vec{x} \} = \text{Nul}(\lambda I_n - A).
\]

Proof.

\[
\{ \vec{x} | A \vec{x} = \lambda \vec{x} \} = \{ \vec{x} | \lambda \vec{x} - A \vec{x} = \vec{0} \} = \{ \vec{x} | (\lambda I_n) \vec{x} - A \vec{x} = \vec{0} \} = \{ \vec{x} | (\lambda I_n - A) \vec{x} = \vec{0} \} = \text{Nul}(\lambda I_n - A).
\]

\( \square \)

Theorem 4. Let \( \lambda \) be an eigenvalue of a matrix \( A \). Then all the \( \lambda \)-eigenvectors of \( A \) together with the zero vector form a subspace, which is called the eigenspace of \( A \) corresponding to \( \lambda \) (or the \( \lambda \)-eigenspace).

Proof. Since the set of all \( \lambda \)-eigenvectors together with \( \vec{0} \) is exactly \( \{ \vec{x} | A \vec{x} = \lambda \vec{x} \} = \text{Nul}(\lambda I_n - A) \), which is of course a subspace.

\( \square \)

Example: Textbook P.304.
**Question?** How to compute the eigenvalues?

**Theorem 5.** Let $A$ be an $n \times n$ square matrix. Then the following statements are equivalent:
1. A scalar $\lambda$ is an eigenvalue of $A$.
2. $\text{Nul}(\lambda I_n - A) \neq \{0\}$.
3. $\dim \text{Nul}(\lambda I_n - A) \geq 1$.
4. $\lambda I_n - A$ is not invertible.
5. $\det(\lambda I_n - A) = 0$.

*Proof.* Since $\lambda$ is an eigenvalue of $A$ iff $\text{Nul}(\lambda I_n - A)$ contains a non-zero vector iff $\text{Nul}(\lambda I_n - A) \neq \{0\}$, so $1. \iff 2.$, and $2. \iff 3. \iff 4. \iff 5.$ is straightforward.

Example: Find all the eigenvalues of $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ and the corresponding eigenspaces.

*Solution.* Suppose that $\lambda$ is an eigenvalue of $A$. Then by Theorem 4, $\det(\lambda I_n - A) = 0$, that is $\det \begin{pmatrix} \lambda - 3 & 1 \\ -2 & \lambda \end{pmatrix} = 0$. By expanding this determinant, we get a quadratic equation $\lambda^2 - 3\lambda + 2 = 0$ in $\lambda$. By solving this equation, we obtain all the eigenvalues of $A$ are 1 and 2.

The 1–eigenspace is $\text{Nul}(I_2 - A)$, that is the solution set of the homogenous equation $(I_2 - A)\vec{x} = \vec{0}$. By solving this equation, we obtain the 1–eigenspace is $\text{Span}\{(1 \ 2)^T\}$.

Similarly, we can find the 2–eigenspace, which is $\text{Span}\{(1 \ 1)^T\}$.

By the Theorem 5, we can easily have:

**Theorem 6.** The eigenvalues of a triangular matrix are the entries on its main diagonal.

Motivated by the above example of finding eigenvalues, we have

**Definition 7** (Characteristic Equation). Let $A$ be a $n \times n$ square matrix. The characteristic equation of $A$ is defined to be $\det(\lambda I_n - A) = 0$. 


Remarks: The eigenvalues of matrix $A$ are precisely the roots of $c_A(\lambda)$ in the set of scalars. Hence, the number of distinct eigenvalues must be less than $n$.

Example: Textbook P.313.

Example: Find all the eigenvalues of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the corresponding eigenspaces over $\mathbb{R}$ and $\mathbb{C}$ respectively.

Solution. 1. $A \in \mathbb{R}^{2 \times 2}$.

The characteristic polynomial $c_A(\lambda) = \det \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix} = \lambda^2 + 1$. Since $c_A(\lambda)$ has no real root, so no eigenvalue exists in $\mathbb{R}$.

2. $A \in \mathbb{C}^{2 \times 2}$.

In this case, $c_A$ has roots $i$ and $-i$, which are all eigenvalues of $A$. The $i$-eigenspace is $\text{Nul}(iI_2 - A)$, which can be computed by solving equation $(iI_2 - A)x = 0$ and the result is $\text{Span}\{(i 1)^T\}$. Similarly, we can obtain the $(-i)$-eigenspace $\text{Span}\{(-i 1)^T\}$.

This example illustrates that the existence of eigenvalues depends on the set of scalars (which is in fact a field).

**Summary of the procedure of computing eigenvalues.**

For a given square matrix $A$, its eigenvalues and the corresponding eigenspaces are computed as follows:

1. Expand $c_A(\lambda) = \det(\lambda I_n - A)$;
2. Solve $c_A(\lambda) = 0$ in the set of scalars ($\mathbb{R}$ or $\mathbb{C}$ in this course) to obtain all the eigenvalues, say $\lambda_1, \ldots, \lambda_r$.
3. For each $i = 1, \ldots, r$, solve the homogenous equation $(\lambda_i I_n - A)x = 0$ and the solution set is the $\lambda_i$-eigenspace.
4 Diagonalization

We are now considering a special matrix factorization for some square matrix using eigenvectors and eigenvalues, which has the following forms

\[ A = PDP^{-1}. \] (6)

where \( P \) is an invertible matrix and \( D \) is a diagonal matrix. This factorization enables us to compute \( A^k \) by

\[ A^k = PD^kP^{-1}. \] (7)

We are now formally introducing the diagonalization processing.

**Definition 8 (Similarity).** Two square matrix \( A \) and \( B \) are similar if there is an invertible matrix \( P \) such that \( P^{-1}AP = B \), or say equivalently \( A = PB^{-1}P \). Changing \( A \) into \( P^{-1}AP \) is called a similarity transformation.

**Definition 9 (Diagonalizable Matrix).** Let \( A \) be a square matrix. \( A \) is said to be diagonalisable if \( A \) is similar to a diagonal matrix \( D \) or equivalently, there exists an invertible matrix \( P \) such that \( P^{-1}AP \) is diagonal.

**Theorem 10.** If two \( n \times n \) matrices \( A \) and \( B \) are similar, they have the same characteristic polynomial and hence the same eigenvalues.

**Proof.** Since \( A \) and \( B \) are similar, there exists an invertible matrix \( P \) such that \( B = P^{-1}AP \). Therefore, we have

\[ B - \lambda I = P^{-1}AP - \lambda P^{-1}IP = P^{-1}(A - \lambda I)P, \]

So

\[
\begin{align*}
\det B &= \det(P^{-1}(A - \lambda I)P) \\
&= \det P^{-1} \det(A - \lambda I) \det P \\
&= \det P^{-1} \det P \det(A - \lambda I) \\
&= \det(P^{-1}P) \det(A - \lambda I) \\
&= \det I \det(A - \lambda I) \\
&= \det(A - \lambda I). \quad (8)
\end{align*}
\]
Not all square matrices are diagonalisable, then we have the following question:

**Question:** When can a square matrix $A$ be diagonalized?

**Theorem 11.** Let $A \in \mathbb{R}^{n \times n}$. $A$ is diagonalisable iff there exists a basis $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ such that $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent eigenvectors of $A$.

**Proof.** We prove the theorem by the following steps:

**STEP 1.** Suppose that $A$ is diagonalisable. Then there exists an invertible matrix $P = (\vec{v}_1 \cdots \vec{v}_n)$ and a diagonal matrix $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ such that $AP = PD$. So

$$\begin{align*}(A\vec{v}_1 \cdots A\vec{v}_n) &= AP = PD = (\lambda_1 \vec{v}_1 \cdots \lambda_n \vec{v}_n).\end{align*}$$

So for all $i$, $\vec{v}_i$ is a $\lambda_i$–eigenvector of $A$, also the invertibility of $P$ ensures that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ forms a basis.

**STEP 2.** Conversely, let $P = (\vec{v}_1 \cdots \vec{v}_n)$. For $i = 1, \ldots, n$, let $\lambda_i$ be the eigenvalue with eigenvector $\vec{v}_i$. Thus we have

$$AP = P\text{diag}(\lambda_1, \ldots, \lambda_n).$$

Since the eigenvectors are linearly independent, $P$ is invertible, thus we have:

$$A = P\text{diag}(\lambda_1, \ldots, \lambda_n)P^{-1}.$$

**Remark:** The basis $\vec{v}_1 \cdots \vec{v}_n$ in the above theorem is called an **eigenvector basis**.

**Corollary 12.** If $A \in \mathbb{R}^{n \times n}$ has $n$ distinct eigenvalues then $A$ is diagonalisable.

**Proof.** By **Theorem 2**, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent, where for $i = 1, \ldots, n$, $\vec{v}_i$ is some $\lambda_i$–eigenvector. So $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis and $A$ is diagonalisable by by **Theorem 9**.

\[ \square \]
Question: Can we still perform diagonalization when not all eigenvalues are distinct?

Theorem 13. Let \( \lambda_1, \ldots, \lambda_r \) be distinct eigenvalues of \( A \in \mathbb{R}^{n \times n} \). For \( i = 1, \ldots, r \), let \( S_i = \{ \vec{v}_{i1}, \ldots, \vec{v}_{in_i} \} \) be a linearly independent set of \( \lambda_i \)-eigenspace. Then \( S = \bigcup_{i=1}^{r} S_i \) is linearly independent.

Proof. For \( i = 1, \ldots, r \), let \( c_{i1}, \ldots, c_{in_i} \) be scalars and \( \vec{w}_i = c_{i1} \vec{v}_{i1} + \cdots + c_{in_i} \vec{v}_{in_i} \). Suppose that \( \vec{w}_1 + \cdots + \vec{w}_r = \vec{0} \). We need to show \( c_{i1} = \cdots = c_{in_i} = 0 \) for all \( i = 1, \ldots, r \). Now for \( i = 1, \ldots, r \), \( \vec{w}_i \) is either \( \vec{0} \) or a \( \lambda_i \)-eigenvector. But \( \{ \vec{w}_1, \ldots, \vec{w}_r \} \) is linearly dependent, which forces that \( \vec{w}_1 = \cdots = \vec{w}_r = \vec{0} \), otherwise contradicting the result of Theorem 2. So for all \( i = 1, \ldots, r \), \( c_{i1} = \cdots = c_{in_i} = 0 \) since \( S_i \) is linearly independent.

Corollary 14. Let \( \lambda_1, \ldots, \lambda_r \) be all distinct eigenvalues of \( A \in \mathbb{R}^{n \times n} \) and \( n_i \) the dimension of \( \lambda_i \)-eigenspace for \( i = 1, \ldots, r \). Then \( A \) is diagonalisable iff \( \sum_{i=1}^{r} n_i = n \).

Proof. Suppose \( A \) is similar to \( D \), where \( D \) is diagonal. Then it is not hard to see that \( A \) and \( D \) have the same set of eigenvalues and the dimensions of the corresponding eigenspaces are equal (Exercise). But for \( D \), hence for \( A \), we have \( \sum_{i=1}^{r} n_i = n \).

Conversely, suppose that \( \mathcal{B}_i \) is a basis of \( \lambda_i \)-eigenspace for \( i = 1, \ldots, r \). Then \( \mathcal{B} = \bigcup_{i=1}^{r} \mathcal{B}_i \) is linearly independent by Theorem 13. Also \( |\mathcal{B}| = \sum_{i=1}^{r} n_i = n \). So \( \mathcal{B} \) is a basis consisting of eigenvectors. So \( A \) is diagonalisable by Theorem 11.

According to Corollary 14, the following procedure can be used to determine whether or not a given \( A \in M_n \) is diagonalisable and in the affirmative case, a matrix \( P \) such that \( P^{-1}AP \) is diagonal is computed.

Step 1: Compute all eigenvalues \( \lambda_1, \ldots, \lambda_r \) of \( A \).

Step 2: For each \( i = 1, \ldots, r \), compute a basis \( \mathcal{B}_i \) of the \( \lambda_i \)-eigenspace.

Step 3: If \( \sum_{i=1}^{r} |\mathcal{B}_i| \neq n \), then \( A \) is not diagonalisable. Otherwise, the matrix \( P = (\vec{v}_1 \cdots \vec{v}_n) \), where \( \{ \vec{v}_1, \ldots, \vec{v}_n \} = \bigcup_{i=1}^{r} \mathcal{B}_i \), satisfies that \( P^{-1}AP \) is diagonal.

Example: Textbook P.324.
4.1 A Linear Transformation View of Diagonalization

**Objective:** We aim to understand the diagonalization of matrix $A$ from the linear transformation view of point (在本节，我们尝试利用线性变换解释对矩阵A的对角话操作).

**STEP 1: Linear Transformation between Vector Space $V$ and $W$**

Let $T$ be the linear transformation between $V$ and $W$. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n\}$ be a basis of vector space $V$. Let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \cdots, \vec{c}_m\}$ be a basis of vector space $W$.

Then any $\vec{x} \in V$ can be represented by a linear combination of $\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n$ with weight $r_1, r_2, \cdots, r_n$ as follows:

$$\vec{x} = r_1\vec{b}_1 + \cdots + r_n\vec{b}_n = B[\vec{x}]_\mathcal{B}.$$  \hspace{1cm} (9)

where $B = [\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n]$ and $[\vec{x}]_\mathcal{B} = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$. Then by the definition of linear transformation, we have

$$T(\vec{x}) = T(r_1\vec{b}_1 + \cdots + r_n\vec{b}_n) = r_1T(\vec{b}_1) + \cdots + r_nT(\vec{b}_n)$$  \hspace{1cm} (9)

So, the coordinate of $T(\vec{x})$ under basis $\mathcal{C}$ in $W$ is

$$[T(\vec{x})]_\mathcal{C} = r_1[T(\vec{b}_1)]_\mathcal{C} + \cdots + r_n[T(\vec{b}_n)]_\mathcal{C} = M[\vec{x}]_\mathcal{B},$$  \hspace{1cm} (10)

where $M = ([T(\vec{b}_1)]_\mathcal{C}, \cdots, [T(\vec{b}_n)]_\mathcal{C})$.

We call matrix $M$ as the matrix for $T$ relative to the bases $\mathcal{B}$ to $\mathcal{C}$.

**STEP 2: Linear Transformation from $V$ to $V$ (i.e. $W = V$).** In this case, the $M$ matrix is called the matrix for $T$ relative to $\mathcal{B}$, or the $\mathcal{B}$-matrix for $T$. We re-denote this $\mathcal{B}$-matrix by $[T]_\mathcal{B}$ as follows:

$$[T]_\mathcal{B} = ([T(\vec{b}_1)]_\mathcal{B}, \cdots, [T(\vec{b}_n)]_\mathcal{B}).$$

Then, for all $\vec{x} \in V$, we have

$$[T(\vec{x})]_\mathcal{B} = [T]_\mathcal{B}[\vec{x}]_\mathcal{B}$$  \hspace{1cm} (11)

**STEP 3: Diagonal Matrix Representation.** Now, we reach the main result of this part.
Theorem 15. Suppose two square matrices $A$ and $C$ are similar, that is there exists an invertible matrix $P$ such that $A = PCP^{-1}$, where $C$ is $n \times n$ matrix. Let $\mathcal{B}$ be the basis for $\mathbb{R}^n$ formed from the columns of $P$, then $C$ is the $\mathcal{B}-$ matrix for the transformation $T : \vec{x} \rightarrow A\vec{x}$.

Proof. Let $P = [\vec{b}_1, \cdots, \vec{b}_n]$, so $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$. Note that
\[
\vec{x} = P[x]_\mathcal{B} \rightarrow P^{-1}\vec{x} = [\vec{x}]_\mathcal{B}.
\]
Also, since we are discussing the linear transformation in $\mathbb{R}^n$, so we have
\[
T(\vec{x}) = A\vec{x}.
\]
Now we investigate the form of $\mathcal{B}-$matrix. That is
\[
[T]_\mathcal{B} = ([T(\vec{b}_1)]_\mathcal{B}, \cdots, [T(\vec{b}_n)]_\mathcal{B}) = ([A\vec{b}_1]_\mathcal{B}, \cdots, [A\vec{b}_n]_\mathcal{B}) = (P^{-1}A\vec{b}_1, \cdots, P^{-1}A\vec{b}_n) = P^{-1}A(\vec{b}_1, \cdots, \vec{b}_n) = P^{-1}AP = C.
\]
(12)

Remark. When $C = D$ in the last theorem, it means diagonalizing $A$ amounts to finding a diagonal matrix representation of $T : \vec{x} \rightarrow A\vec{x}$.

5 Complex Eigenvalues

Actually all our preceding introduction can be straightforwardly extended to the complex domain, by changing $\mathbb{R}^n$ to $\mathbb{C}^n$ and changing $\mathbb{R}^{n \times n}$ to $\mathbb{C}^{n \times n}$. When the eigenvalue $\lambda$ is a complex value, we call it the complex eigenvalue and its corresponding vect

\[
\text{Rhombicuboctahedron, by Da Vinci}
\]