15. Dimension and Rank

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1 What Do You Learn from This Note

We still observe the unit vectors we have introduced in Chapter 1:

\[ \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \] (1)

We know the above are the basis (specially the standard basis) of \( \mathbb{R}^3 \). However, we still have to answer the following question:

**Question:** Why there are three basis in \( \mathbb{R}^3 \)?

**Basic Concept:** dimension(维数), rank(秩)
2 Dimension

**Theorem 1.** Let $V$ be a vector space with basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$. Then any subset of $V$ containing more than $n$ vectors is linearly dependent.

**Proof.** Let $p \in \mathbb{N}$ and $p > n$. Assume that $\{\vec{u}_1, \ldots, \vec{u}_p\} \subseteq V$. As $\mathcal{B}$ is the basis, so we have for each $\vec{u}_i$ there exists a coefficient vector $\vec{a}_i \in \mathbb{R}^n$ such that

$$\vec{u}_i = B\vec{a}_i, \quad B = [\vec{b}_1, \ldots, \vec{b}_n].$$

Let $A = [\vec{a}_1, \ldots, \vec{a}_p], U = [\vec{u}_1, \ldots, \vec{u}_p]$. Then

$$U = BA.$$ 

The if there is a series of weight $c_1, \ldots, c_p$ such that $c_1\vec{u}_1 + \cdots + c_p\vec{u}_p = \vec{0}$, that is

$$U\vec{c} = \vec{0},$$

then we have

$$BA\vec{c} = \vec{0}.$$ 

As the number of rows is smaller than the number of columns of $BA$, so that is there are non-pivot columns in $BA$. This leads to the matrix equation $BA\vec{c}$ has non-trivial solution. So there exists non-zeros $\vec{c}$ such that $U\vec{c} = \vec{0}$. That is $\vec{u}_1, \ldots, \vec{u}_p$ are linearly dependent.

**Theorem 2.** Let $V$ be a vector space with bases $\mathcal{B}, \mathcal{C}$ of sizes $m, n \in \mathbb{N}$ respectively. Then $m = n$. That is every basis of $V$ has the same size.

**Proof.** Since $\mathcal{C}$ is linearly independent, by **Theorem 1**, we must have $m \geq n$. Similarly, $n \geq m$. So $m = n$. 

**Definition 3** (dimension). Let $V$ be a vector space with a finite subset as its basis. Then the size of its basis is called the dimension of $V$ and is denoted by $\dim V$, and we say $V$ is finite-dimensional. Otherwise, if $V$ can not be spanned by a finite set then $V$ is infinite-dimensional.

Remark:
(1) $\dim \mathbb{R}^n = n$.
(2) The space with all polynomials (多项式空间) is infinite-dimensional.
3 Subspace & Dimension

**Theorem 4.** Let \( V \) be a vector space of dimension \( n \in \mathbb{N} \). Then any linearly independent subset \( \{\vec{v}_1, \ldots, \vec{v}_m\} \) of \( V \) can be extended to a basis of \( V \).

**Proof.** We prove this theorem by the following steps:

1. When \( m = n \), because if Span\( \{\vec{v}_1, \ldots, \vec{v}_m\} = V \) then \( \{\vec{v}_1, \ldots, \vec{v}_m\} \) is a basis of \( V \).

2. When \( m < n \) Span\( \{\vec{v}_1, \ldots, \vec{v}_m\} \subset V \) and we can pick \( \vec{v}_{m+1} \in V - \text{Span}\{\vec{v}_1, \ldots, \vec{v}_m\} \). We claim that \( \{\vec{v}_1, \ldots, \vec{v}_m, \vec{v}_{m+1}\} \) is linearly independent. Assume this is not the case, then there exist \( c_1, \ldots, c_m, c_{m+1} \) which are not all zero such that \( c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m + c_{m+1} \vec{v}_{m+1} = 0 \).
   
   If \( c_{m+1} = 0 \) then \( \{\vec{v}_1, \ldots, \vec{v}_m\} \) is linearly dependent, otherwise \( \vec{v}_{m+1} \in \text{Span}\{\vec{v}_1, \ldots, \vec{v}_m\} \). Both cases contradict the assumptions we made. So \( \{\vec{v}_1, \ldots, \vec{v}_m, \vec{v}_{m+1}\} \) is linearly independent.

3. If \( \{\vec{v}_1, \ldots, \vec{v}_m, \vec{v}_{m+1}\} \) spans \( V \) then we obtain a basis, otherwise repeat this process again and finally, we must obtain a linearly independent set containing \( \{\vec{v}_1, \ldots, \vec{v}_m\} \) spans \( V \), that is a basis.

**Theorem 5.** Let \( H \) be a subspace of a finite-dimension vector space \( V \). Any linearly independent set in \( H \) can be expanded, if necessary, to a basis for \( H \). Also, \( H \) is finite-dimensional and

\[
\dim H \leq \dim V
\]

**Theorem 6.** Let \( V \) be a vector space of dimension \( p \geq 1 \). Any linearly independent set of exactly \( p \) elements in \( V \) is automatically a basis for \( V \). Any set of exactly \( p \) elements that spans \( V \) is automatically a basis for \( V \).

**Proof.** By theorem 4, any linearly independent set \( S \) of size \( q \leq p \) can be expanded to the basis of \( V \). This implies, if \( q = p \), then \( S \) must be the basis of \( V \).

**Theorem 7.** The dimension of NuA is the number of free variables in the equation \( A\vec{x} = 0 \), and the dimension of ColA is the number of pivot columns in \( A \).
4 Rank of a Matrix $A$

4.1 Row Space

**Definition 8 (Row Space).** If $A$ is an $m \times n$ matrix. The set of all linear combinations of the row vectors is called the row space of $A$, denoted by $\text{Row}A$.

From another point of view, the rows of $A$ are identical to the columns of $A^T$, so the row space of $A$ can also be written as $\text{Col}A^T$.

**Theorem 9.** If two $m \times n$ matrices $A$ and $B$ are row equivalent, their row spaces are the same.

**Proof.** Since $A$ and $B$ are row equivalent, there exists an invertible matrix $G$ such that

$$B = GA.\tag{1}$$

That is

$$B^T = A^T G^T = A^T F, \quad F = G^T.\tag{2}$$

By using the matrix partition theory, let $A^T = [\text{Col}_1 A^T, \ldots, \text{Col}_m A^T]$, $F = [\text{Col}_1 F, \ldots, \text{Col}_m F]$, then

$$\begin{align*}
[\text{Col}_1 B^T, \ldots, \text{Col}_i B^T, \ldots, \text{Col}_m A^T] \\
= B^T \\
= A^T F \\
= [A^T \text{Col}_1 F, \ldots, A^T \text{Col}_i F, \ldots, A^T \text{Col}_m F].
\end{align*}$$

That is

$$\text{Col}_i B^T = A^T \text{Col}_i F.\tag{3}$$

Since $(\text{Row}_i A)^T = \text{Col}_i A^T$ and $(\text{Row}_i B)^T = \text{Col}_i B^T$, so

$$\text{Row}_i B = [F]_{1i} \text{Row}_i A + \cdots + [F]_{mi} \text{Row}_m A.$$ 

That is $\text{Row}B \subseteq \text{Row}A$.

Conversely, by changing the roles of $A$ and $B$ in the above, we can also have $\text{Row}B \subseteq \text{Row}A$. So, $\text{Row}B = \text{Row}A$.

**Theorem 10.** If two $m \times n$ matrices $A$ and $B$ are row equivalent, and $B$ is in echelon form, then the nonzero rows of $B$ form a basis for the row space of $A$ as well as for that of $B$.
4.2 Definition and Properties of Rank

**Definition 11** (column & row rank). Let $A$ be a $m \times n$ matrix. The column rank of $A$ is defined to be $\dim \text{Col}(A)$ and the row rank of $A$ to be $\dim \text{Row}(A)$.

Remark: The dimension of the null space is sometimes called the **nullity** of $A$.

**Theorem 12** (Rank Theorem). The dimensions of the column space and the row space of an $m \times n$ matrix $A$ are equal. Also, it holds that:

\[ \text{rank} A + \dim \text{Nul} A = n. \] (4)

*Proof.* We know

1. rank$A$ is the number of pivot columns in $A$. That is if $B$ is an echelon form of $A$, i.e. rank$A$ is the number of pivot positions in $B$.
2. Each pivot position corresponds to a nonzero row, and these rows form a basis for the row space of $A$, so the rank of $A$ is also the dimension of the row space.
3. The dimension of Nul$A$ equals the number of free variables in the equation $A\vec{x} = \vec{0}$. That is the dimension of Nul$A$ is the number of columns of $A$ that are not pivot columns.
4. Number of pivot columns + number of non-pivot columns = number of columns, i.e.

\[ \text{rank} A + \dim \text{Nul} A = n. \]
5 Computation of Null Space and Rang Space

We are now discussing how to compute bases of \( \text{Col}(A) \) and \( \text{Nul}(A) \) given \( A \in \mathbb{R}^{m \times n} \).

**Computation of Column Space:** Let \( A = (\vec{a}_1 \ldots \vec{a}_n) \in \mathbb{R}^{m \times n} \). Then there exists an invertible matrix \( B \in \mathbb{R}^{m \times m} \) such that \( BA \) is in REF. Let \( B\vec{a}_i_1, \ldots, B\vec{a}_i_r \) be all the pivot columns of \( BA \). Then it is obvious that \( \{B\vec{a}_i_1, \ldots, B\vec{a}_i_r\} \) is a basis of \( \text{Col}(BA) \). Since \( B \) is invertible and by Theorem 15.4, \( \{\vec{a}_i_1, \ldots, \vec{a}_i_r\} \) forms a basis of \( \text{Col}(A) \), that is the set of pivot columns of \( A \) is a basis of \( \text{Col}(A) \).

**Computation of Null Space:** We also know that \( \text{Nul}(A) \) is exactly the solution set of the equation \( Ax = \vec{0} \). So by solving \( Ax = \vec{0} \) using Row Reduction Algorithm, we obtain \( \vec{v}_1, \ldots, \vec{v}_{n-r} \in \text{Nul}(A) \), such that \( \text{Nul}(A) = \{x_{j_1}\vec{v}_1 + \cdots + x_{j_{n-r}}\vec{v}_{n-r} \mid x_{j_1}, \ldots, x_{j_{n-r}} \in \mathbb{R}\} = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{n-r}\} \), where \( x_{j_1}, \ldots, x_{j_{n-r}} \) correspond to the free variables of \( Ax = \vec{0} \). But by the Rank Theorem, \( \dim \text{Nul}(A) = n - \dim \text{Col}(A) = n - r \). Thus \( \{\vec{v}_1, \ldots, \vec{v}_{n-r}\} \) is a basis of \( \text{Nul}(A) \) by Theorem 5.

6 Rank & Matrix Inverse

Theorem 13 (The Invertible Matrix Theorem). Let $A$ be an $n \times n$ matrix. The following statements are each equivalent to the statement that $A$ is an invertible matrix:

1. The columns of $A$ form a basis of $\mathbb{R}^n$;
2. $\text{Col} A = \mathbb{R}^n$;
3. $\dim \text{Col} A = \mathbb{R}^n$;
4. $\text{rank} A = n$;
5. $\text{Nul} A = \{0\}$;
6. $\dim \text{Nul} A = 0$;

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