

LECTURE NOTE ON LINEAR ALGEBRA

5. LINEAR INDEPENDENCE

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1 What Do You Learn from This Note

Learn the minimal set for describing a vector space. This also helps understand the solution set of a linear system

Basic concept: linear independence(线性独立)

2 Linear Independence(线性独立)

Let S be a set of vectors, then $W = \text{Span}\{S\}$ is the set (or subspace (子空间) introduced later) spanned by S . An interesting question is whether S is a minimal set for describing the set W (一个有趣的问题是, S 是不是张成数据集 W 的最小向量集). Here, S is a minimal set for describing W means for any proper subset (真子集) S' of S (i.e. $S' \subset S$), $\text{Span}\{S'\}$ is also a proper subset of W).

Example: Let $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then it is easy to see that $\text{Span}\{\vec{e}_1, \vec{e}_2, \vec{v}\} = \text{Span}\{\vec{e}_1, \vec{e}_2\} = \mathbb{R}^2$. So the set $\{\vec{e}_1, \vec{e}_2, \vec{v}\}$ is not minimal. However, $\{\vec{e}_1, \vec{e}_2\}$ is minimal since \mathbb{R}^2 can not be spanned by any one of $\{\vec{e}_1\}$, $\{\vec{e}_2\}$ and \emptyset which are all the proper subsets of $\{\vec{e}_1, \vec{e}_2\}$.

DEFINITION 1 (linear independence). Vectors $\vec{v}_1, \dots, \vec{v}_n$ (or set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$) are said to be linearly independent (线性独立) if and only if (iff, 当且仅当)

$$c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$$

implies

$$c_1 = \dots = c_n = 0.$$

Remarks:

1. Vectors $\vec{v}_1, \dots, \vec{v}_n$ are said to be linearly dependent if they are not linearly independent.

2. If one of $\vec{v}_1, \dots, \vec{v}_n$ is zero, then $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent. To see this, with loss of generality, we may let $\vec{v}_1 = \vec{0}$. Then

$$1 \cdot \vec{0} + 0 \cdot \vec{v}_2 + \dots + 0 \cdot \vec{v}_n = \vec{0},$$

namely $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent.

3. From the perspective of equations, vectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent iff $(0, \dots, 0)$ is the only solution of the vector equation

$$x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{0}$$

iff $\vec{0}$ is the only solution of the matrix equation

$$(\vec{v}_1 \ \dots \ \vec{v}_n)\vec{x} = \vec{0}.$$

So whether vectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent can be determined by solving matrix equation.

Examples: Textbook P.65, P.66(板书).

THEOREM 2. The columns of a matrix $A = (\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$ are linearly independent if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution (平凡解, i.e. \vec{x} 为零向量)

Proof: Exercise.

3 Analysis of Linear Independence

The next theorem answers the question of minimal spanning set posted at the beginning, which gives a characterization of linearly dependent set.

THEOREM 3. Vectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent iff for some $i = 1, \dots, n$, $\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}$, in other words, \vec{v}_i is a linear combination of $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n$.

Proof. Suppose that $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent. By definition, vector equation

$$x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{0}$$

has a non-zero solution (c_1, \dots, c_n) . Let i be such that $c_i \neq 0$. Then we have $c_1\vec{v}_1 + \dots + c_i\vec{v}_i + \dots + c_n\vec{v}_n = \vec{0}$, which gives $\vec{v}_i = \left(-\frac{c_1}{c_i}\right)\vec{v}_1 + \dots + \left(-\frac{c_{i-1}}{c_i}\right)\vec{v}_{i-1} + \left(-\frac{c_{i+1}}{c_i}\right)\vec{v}_{i+1} + \dots + \left(-\frac{c_n}{c_i}\right)\vec{v}_n$, i.e. $\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}$.

Conversely, suppose that for some $i = 1, \dots, n$, we have

$$\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}.$$

Then $\vec{v}_i = c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \dots + c_n\vec{v}_n$ for some $c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n \in \mathbb{R}$. Thus we have

$$c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1} + (-1)\vec{v}_i + c_{i+1}\vec{v}_{i+1} + \dots + c_n\vec{v}_n = \vec{0},$$

which implies that $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent. □

So by this theorem, we have S is a minimal set spanning $\text{Span}\{S\}$ iff S is a linearly independent set, or equivalently, S is not a minimal set spanning $\text{Span}\{S\}$ iff S is linearly dependent, which answer the question at the beginning.

The next theorem tells you that the number of elements of a linearly independent set of \mathbb{R}^m can not be larger than m .

THEOREM 4. Vectors $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^m$ are linearly dependent if $n > m$.

Proof. Let us consider the matrix equation $(\vec{v}_1 \cdots \vec{v}_n)x = \vec{0}$. The augmented matrix of this equation is $(\vec{v}_1 \cdots \vec{v}_n \vec{0})$, which is a $m \times (n + 1)$ matrix. Since $n > m$ means the number of variables is larger than the number of equations, by the previous discussion on solution set of system of linearly equations, this matrix equation has infinite many solutions. So v_1, \dots, v_n are linearly dependent. \square

