Lecture Note on Linear Algebra
4. Matrix Equations

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1 What Do You Learn from This Note

The product between a matrix and a vector. We will also discuss something about the existence of solutions of a linear system (线性方程组解的存在性).
Basic concept: matrix equation(矩阵方程)

2 Matrix Equation

Definition 1 (product of matrix and vector(矩阵与向量之间的乘法)). Let $A = (\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n)$ be an $m \times n$ matrix ($\vec{a}_i \in \mathbb{R}^m$)and $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$.
Then define the product of A and c as

$$A\vec{c} = (\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n,$$

which is the linear combination of $\vec{a}_1, \ldots, \vec{a}_n$ with coefficients $c_1, \ldots, c_n$.

Example:

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 2 \\ -5 \end{pmatrix} + 7\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Remarks:

1. $A\vec{c}$ is defined (有定义的) only if the number of columns of A is equal to the number of entries of $\vec{c}$.(就是矩阵A的列数与列向量$\vec{c}$的维数一致)

2. Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$. Then

$$A\vec{c} = c_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} c_1 a_{11} + \cdots + c_n a_{1n} \\ \vdots \\ c_1 a_{m1} + \cdots + c_n a_{mn} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n c_j a_{1j} \\ \vdots \\ \sum_{j=1}^n c_j a_{mj} \end{pmatrix}$$

Definition 2 (matrix equation(矩阵方程)). Let A be an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$. Also let x denote an undetermined vector in $\mathbb{R}^n$. Then we call

$$A\vec{x} = \vec{b}$$

a matrix equation in x. A solution of $A\vec{x} = \vec{b}$ is a vector $s = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \in \mathbb{R}^n$ satisfying $A\vec{s} = \vec{b}$.

Remark. If we write $A = (\vec{a}_1 \ \cdots \ \vec{a}_n)$, $\vec{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$, then we have

$$A\vec{s} = (\vec{a}_1 \ \cdots \ \vec{a}_n) \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = s_1\vec{a}_1 + \cdots + s_n\vec{a}_n = \vec{b}.$$

In other words, $s = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$ is a solution of $A\vec{x} = \vec{b}$ iff $(s_1, \cdots, s_n)$ is a solution of the vector equation $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$. So we have the following fact:

$$\begin{cases} a_{11}x_1 & + \cdots & + a_{1n}x_n & = & b_1 \\ \cdots & & & & \\ a_{m1}x_1 & + \cdots & + a_{mn}x_n & = & b_m \end{cases} \Leftrightarrow x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b} \Leftrightarrow A\vec{x} = \vec{b},$$

that is, they have the same solution set.

Now we can view a system of linear equations in three different but equivalent ways:

1. A matrix equation
2. A vector equation
3. A collection of linear equations

We finally present some properties of the product of matrix and vector:

THEOREM 3. Let A be an $m \times n$ matrix and $\vec{u}, \vec{v} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. Then
1. $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$;
2. $A(\lambda\vec{u}) = \lambda(A\vec{u})$.

Proof. Suppose that $A = (\vec{a}_1 \ \cdots \ \vec{a}_n)$, $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$.

1.

$$\begin{aligned} A(\vec{u}+\vec{v}) &= (u_1+v_1)\vec{a}_1 + \cdots + (u_n+v_n)\vec{a}_n \\ &= u_1\vec{a}_1 + v_1\vec{a}_1 + \cdots + u_n\vec{a}_n + v_n\vec{a}_n \\ &= (u_1\vec{a}_1 + \cdots + u_n\vec{a}_n) + (v_1\vec{a}_1 + \cdots + v_n\vec{a}_n) \\ &= A\vec{u} + A\vec{v} \end{aligned}$$

2.

$$\begin{aligned} A(\lambda\vec{u}) &= (\lambda u_1)\vec{a}_1 + \cdots + (\lambda u_n)\vec{a}_n \\ &= \lambda(u_1\vec{a}_1) + \cdots + \lambda(u_n\vec{a}_n) \\ &= \lambda(u_1\vec{a}_1 + \cdots + u_n\vec{a}_n) \\ &= \lambda(A\vec{u}) \end{aligned}$$

□

3 A Theorem on Consistent (相容的) Solution for Matrix Equation

The next theorem tells under what conditions of A, the solution set of $A\vec{x} = \vec{b}$ is non–empty (consistent) for any $\vec{b}$.

THEOREM 4. Let $A = (\vec{a}_1 \ \cdots \ \vec{a}_n)$ be an $m \times n$ matrix. Then the following statements are equivalent.
1. $\forall \vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution.
2. $\forall \vec{b} \in \mathbb{R}^m$, $\vec{b}$ is a linear combination of $\vec{a}_1, \ldots, \vec{a}_n$.
3. $\mathbb{R}^m = \mathrm{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$.
4. A has a pivot position (主元位置) in every row.

Proof. Proof. We say all the above statements are equivalent means that any one of the statements can be derived from any other one (如果上面四个命题等价，那么其中任何一个命题都可以由其他3个命题之一推出). Statements (命题)1.–3. are obviously equivalent (等价性显而易见) since they are logically equivalent. So our focus on proving the equivalent between statement 1 and statement 4.

Let U be the matrix in REF (reduced echelon form, 简化阶梯形) such that $U \sim A$ (即U与A等价). So there is a series of matrices

$$A = A_0 \xrightarrow{op_1} A_1 \xrightarrow{op_2} \cdots \xrightarrow{op_l} A_l = U,$$

where $op_1, \ldots, op_l$ are elementary row operations. We can perform row reduction (行化简变换) algorithm on $[A \ \vec{b}]$ and obtain a reduced augmented matrix $[U \ \vec{d}]$.

So, if statement 4 is true, there is no pivot in the augmented column, so $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$, and statement 1 is true.

If statement 1 is true, we finally wish to prove statement 4 is true. We reach this claim by contradiction (反正法) Assume that there exits a row of A which has no pivot position. Then, the last row of U must be zero. That is there is no solution for the linear system corresponding to augmented matrix $[U \ \vec{d}]$. Note that the solutions between these two systems (corresponding to A and U respectively) are equivalent, so there is also no solution for the linear system $A\vec{x} = \vec{b}$ too. This yields contradiction to statement 1, so statement 4 is true. (也就是说，通过初等变换后，两个线性系统的解是一样的，所以等价的系统无解，原系统也无解.但这个结论与已知的命题1矛盾，所以在这种情况下假设命题4无解是错的。) $\square$

Reference

David C. Lay. Linear Algebra and Its Applications (3rd edition). Pages 41∼50.

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