1 What Do You Learn from This Note

Recall that we have ever said that $\mathbb{R}^2$, which is the set of plane vectors, is a concrete example of vector space. In this lecture, we will look at some detailed properties of $\mathbb{R}^2$. Further, we will generalize the properties of $\mathbb{R}^2$ to $\mathbb{R}^n$, which is the set of vectors of dimension $n$ over $\mathbb{R}$. We shall see quickly that any system of linear equations is equivalent to a so called vector equation. The study of $\mathbb{R}^n$ will help you to understand the abstract concept of vector spaces which are the major subject studied in linear algebra.

Basic concept: column vector (列向量), linear combination (线性组合), Span (张)

2 Vectors

Definition 1 (vectors (向量) in $\mathbb{R}^2$). A $2 \times 1$ matrix

\[
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix}
\]

is called a column vector (列向量) (or simply a vector) of dimension 2 over $\mathbb{R}$, where $v_1, v_2$ are real numbers. The set of all such vectors is denoted by $\mathbb{R}^2$. Similarly, we can define row vectors.

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. We have the following definitions on $\mathbb{R}^2$:
1. Equality: we say \( \vec{u} \) and \( \vec{v} \) are equal, written \( \vec{u} = \vec{v} \), iff \( u_1 = v_1 \) and \( u_2 = v_2 \).

2. Addition: the vector \( \left( \begin{array}{c} u_1 + v_1 \\ u_2 + v_2 \end{array} \right) \) is called the sum of \( \vec{u} \) and \( \vec{v} \) and is denoted by \( \vec{u} + \vec{v} \).

3. Scalar Multiplication: the vector \( \left( \begin{array}{c} cu_1 \\ cu_2 \end{array} \right) \) is called the scalar multiple of \( \vec{u} \) by scalar \( c \) and is denoted by \( c\vec{u} \).

Example: Let \( \vec{u} = \left( \begin{array}{c} 1 \\ -2 \end{array} \right) \) and \( \vec{v} = \left( \begin{array}{c} 2 \\ -5 \end{array} \right) \), find \( 4\vec{u}, -3\vec{v} \) and \( 4\vec{u} + (-3)\vec{v} \).

Remarks: 1. The vector \( \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \) is called the zero vector and is denoted by \( \vec{0} \).
2. For the sake of simplicity, we normally write \( -\vec{u} \) for \( (-1)\vec{u} \) which is called the additive inverse of \( \vec{u} \) and \( \vec{u} - \vec{v} \) for \( \vec{u} + (-1)\vec{v} \) which is called the difference of \( \vec{u} \) by \( \vec{v} \).

**THEOREM 2.** Let \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2 \) and \( c, d \in \mathbb{R} \). Then
1. \( (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \) (Additive Associativity);
2. \( \vec{0} + \vec{u} = \vec{u} + \vec{0} \) (Additive Identity);
3. \( (-\vec{u}) + \vec{u} = \vec{u} + (-\vec{u}) = \vec{0} \) (Additive Inverse);
4. \( \vec{u} + \vec{v} = \vec{v} + \vec{u} \) (Additive Commutativity);
   (properties 1–4 are called the properties of addition for abelian group (阿贝尔群))
5. \( c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v} \);
6. \( (c + d)\vec{u} = c\vec{u} + d\vec{u} \);
7. \( c(d\vec{u}) = (cd)\vec{u} \);
8. \( 1\vec{u} = \vec{u} \).

**Proof.** Exercise. \( \square \)

Suppose that we have created a coordinate system on a plane. Geometrically, vector \( \vec{v} = \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) \) have 2 interpretations on this plane:

1. A point on the plane, vector \( \vec{v} \) can be interpreted as a point with coordinate \( (v_1, v_2) \). This is a one to one correspondence between points and vectors.
2. A directed segment (线段) (or arrow) with start point \((s_1, s_2)\) and end point \((e_1, e_2)\) such that \(v_1 = e_1 - s_1, v_2 = e_2 - s_2\). Note that this correspondence is one to many (一对多) rather than one to one (一对一).

Parallelogram Rule (平行四边形法则) for addition:

We now define vectors in \(\mathbb{R}^n\) in the same way as in \(\mathbb{R}^2\).

**Definition 3 (vectors in \(\mathbb{R}^n\)).** A \(n \times 1\) matrix

\[
\begin{pmatrix}
  v_1 \\
  \vdots \\
  v_n
\end{pmatrix}
\]

is called a column vector (or simply a vector) of dimension \(n\) over \(\mathbb{R}\), where \(v_1, \ldots, v_n\) are real numbers. The set of all such vectors is denoted by \(\mathbb{R}^n\). (Similarly, we can define row vectors).

Analogous to \(\mathbb{R}^2\), we can define ‘Equality’, ‘Addition’, and ‘Scalar Multiplication’ on \(\mathbb{R}^n\) in a similar way and Theorem 2 also holds for \(\mathbb{R}^n\).

Next, we shall see how to connect vectors in \(\mathbb{R}^n\) to systems of linear equations, we first introduce the following:

### 3 Linear Combinations (线性组合)

**Definition 4 (linear combination (线性组合)).** Given \(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^m\) and \(c_1, c_2, \ldots, c_n \in \mathbb{R}\). Then we can define a new vector

\[
\vec{v} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n = \sum_{i=1}^{n} c_i \vec{v}_i,
\]

which is called a linear combination of \(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\) with coefficients (or weights) \(c_1, c_2, \ldots, c_n\).

**Example:** Let \(\vec{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}\) and \(\vec{v} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}\). Then

\[
4\vec{u} - 3\vec{v} = \begin{pmatrix} 4 - 3 \cdot 2 \\ 4(-2) - 3(-5) \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}
\]
is a linear combination of \( \vec{u} \) and \( \vec{v} \).

Example: Define \( \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \vec{e}_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^m \). Then we have

\[
\begin{pmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_m
\end{pmatrix} = c_1 \vec{e}_1 + \cdots + c_m \vec{e}_m.
\]

This means that any vector in \( \mathbb{R}^m \) is a linear combination of \( \vec{e}_1, \ldots, \vec{e}_m \).

Example: Let \( \vec{a}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \vec{a}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} \) and \( \vec{b} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix} \). Determine whether or not \( \vec{b} \) is a linear combination of \( \vec{a}_1 \) and \( \vec{a}_2 \).

**Solution.** Suppose that \( x_1, x_2 \in \mathbb{R} \) such that \( \vec{b} = x_1 \vec{a}_1 + x_2 \vec{a}_2 \). Then we have

\[
\begin{pmatrix}
  7 \\
  4 \\
  -3
\end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{pmatrix}
\]

By equality of vectors, we have

\[
\begin{cases}
  x_1 + 2x_2 = 7 \\
  -2x_1 + 5x_2 = 4 \\
  -5x_1 + 6x_2 = -3
\end{cases}
\]

So if we regard \( x_1, x_2 \) as unknowns, then \( \vec{b} \) is a linear combination of \( \vec{a}_1 \) and \( \vec{a}_2 \) if and only if the above system of linear equations with augmented matrix \((\vec{a}_1 \ \vec{a}_2 \ \vec{b})\) is consistent.

**Definition 5 (vector equation (向量方程)).** Let \( \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n, \vec{b} \in \mathbb{R}^m \). Then \( x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = \vec{b} \) is called a vector equation with variables \( x_1, x_2, \ldots, x_n \).

From the above example, it is easy to see that the solution set of \( x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = \vec{b} \) is the same as the system of linear equations with
augmented matrix \((\vec{a}_1 \, \vec{a}_2 \, \cdots \, \vec{a}_n \, \vec{b})\).

One of the key ideas in linear algebra is to study the set of all vectors that can be generated or written as a linear combination of a fixed (固定) set \(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^m\) of vectors.

**Definition 6 (Span).** Let \(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^m\). The set of all linear combinations of \(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\) is called the set generated (或 spanned, 张成) by \(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\) and is denoted by \(\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}\), which is a subset of \(\mathbb{R}^m\). We also say \(\vec{w} \in \mathbb{R}^m\) can be generated by \(\vec{v}_1, v_2, \ldots, \vec{v}_n\) if \(\vec{w} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}\).

**Theorem 7.** Let \(W = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}\). Then
1. \(\vec{0} \in W\);
2. \(\vec{v}_i \in W\) for \(i = 1, \ldots, n\);
3. \(\vec{w} \in W\) implies \(\lambda \vec{w} \in W\), where \(\lambda \in \mathbb{R}\);
4. \(\vec{w}_1, \vec{w}_2 \in W\) implies \(\vec{w}_1 + \vec{w}_2 \in W\).

**Proof.** 1. & 2. are obvious.
3. Suppose \(\vec{w} = \sum_{i=1}^{n} c_i \vec{v}_i\). Then \(\lambda \vec{w} = \sum_{i=1}^{n} (\lambda c_i) \vec{v}_i \in W\).
4. Suppose \(\vec{w}_1 = \sum_{i=1}^{n} c_{1i} \vec{v}_i\) and \(\vec{w}_2 = \sum_{i=1}^{n} c_{2i} \vec{v}_i\). Then \(\vec{w}_1 + \vec{w}_2 = \sum_{i=1}^{n} (c_{1i} + c_{2i}) \vec{v}_i \in W\). □

**Reference**