Lecture Note on Linear Algebra

6. Solution Sets of Matrix Equations

Wei-Shi Zheng,
wszheng@ieee.org, 2011

September 20, 2011

1 What Do You Learn from This Note

Given a matrix equation $A\vec{x} = \vec{b}$, where A is an $m \times n$ matrix, we know that it is equivalent to the system of linear equations with augmented matrix $[A\ \vec{b}]$. Hence, the solution set can be determined using the method introduced in the previous lectures. In this lecture, we shall study the structure of the solution set, which is a subset of $\mathbb{R}^n$, in greater detail.

Basic concept：Homogeneous(齐次), trivial solution (平凡解)

2 Homogeneous Matrix Equation(齐次矩阵方程组)

Definition 1 (Homogeneous(齐次的)). *A system of linear equation is said to be homogeneous if it can be written as $A\vec{x} = \vec{0}$, where $\vec{0}$ is the zero vector (零向量) in $\mathbb{R}^m$. $A\vec{x} = \vec{0}$ is called a homogeneous (matrix) equation (齐次矩阵方程).*

Obviously, the zero vector is a solution to the homogeneous linear system and this solution is called trivial solution (平凡解). If there is more solutions rather than the trivial solution to the system, we call the homogeneous equation $A\vec{x} = \vec{0}$ has a nontrivial solution (非平凡解)

In general, the solution set of any homogeneous equation can be expressed as a set spanned by some vectors. Let us illustrate this fact via a concrete example.

Example: Solve the equation $A\vec{x} = \vec{0}$, where

$$A \sim \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that the free variables of this equation are x_2, x_4 and further $x_1 = 2x_2 - x_4$, $x_3 = -8x_4$. So the general solution can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_2 - x_4 \\ x_2 \\ -8x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -8 \\ 1 \end{pmatrix}.$$

Thus, we obtain that $\mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ -8 \\ 1 \end{pmatrix}$ is exactly the solution set of this equation. Moreover, it is easy to verify that $\vec{v}_1$, $\vec{v}_2$ are linearly independent.

注：下面的定理理解上有一定难度。在本节先供参考，以后学到的知识会更容易理解。

THEOREM 2. *Let p denote the number of free variables of a homogeneous equation $A\vec{x} = \vec{0}$.*

- *For $p > 0$, the solution set of this equation can be spanned by p independent vectors, say $\vec{v}_1, \dots, \vec{v}_p$, i.e.*

  $$\textit{Solution Set} = \mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_p\}.$$

  The general solution can be written as

  $$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p,$$

  which is said to be in parametric vector form(参数向量形式).

- *For $p = 0$, i.e. no free variable, $\vec{0}$ is the unique solution. In this case, the solution set can be written as* $\text{Span}\{\vec{0}\}$.

Geometrically, the solution set represents the origin if $p = 0$, a line through the origin if $p = 1$, a subspace of dimension 2 if $p = 2$ (a plane through the origin in the case that we work in $\mathbb{R}^3$) and so on.(这里详见板书进一步解释)

3 Non-homogeneous matrix equations (非齐次矩阵方程)

For the case of non–homogeneous matrix equations, we have the following theorem.

THEOREM 3. Suppose that matrix equation $A\vec{x} = \vec{b}$ has a solution $\vec{s}$. Let S and S_0 be the solution sets of $A\vec{x} = \vec{b}$ and its corresponding homogeneous equation $A\vec{x} = \vec{0}$ respectively. Then we have

$$S = \{\vec{s} + \vec{s}_0 \mid \vec{s}_0 \in S_0\} = \vec{s} + S_0.$$

Proof. (1) Let $t \in S$. Then we have

$$\begin{array}{rcl} A(\vec{t} - \vec{s}) & = & A\vec{t} - A\vec{s} \\ & = & \vec{b} - \vec{b} \\ & = & \vec{0}. \end{array}$$

This means that $\vec{s}_0 = \vec{t} - \vec{s} \in S_0$. So $\vec{t} = \vec{s} + \vec{s}_0 \in \vec{s} + S_0$, i.e. $S \subseteq \vec{s} + S_0$.
(2) On the other hand, Let $\vec{s}_0 \in S_0$. Then we have

$$\begin{array}{rcl} A(\vec{s} + \vec{s}_0) & = & A\vec{s} + A\vec{s}_0 \\ & = & \vec{b} + \vec{0} \\ & = & \vec{b}. \end{array}$$

So $\vec{s} + \vec{s}_0 \in S$, i.e. $\vec{s} + S_0 \subseteq S$.
From (1) and (2), $S = \vec{s} + S_0$. □

So by THEOREM 3, the solution set of equation $A\vec{x} = \vec{b}$ can be written as

$$\vec{s} + \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}.$$

The parametric form of this equation is $\vec{s} + c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$.

Example: Solve the equation $A\vec{x} = \vec{b}$, where

$$(A\ \vec{b}) \sim \begin{pmatrix} 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again, the free variables are x_2, x_4 and further $x_1 = 1 - 2x_2 - x_4$, $x_3 = -1 - 8x_4$. So

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 + 2x_2 - x_4 \\ x_2 \\ -1 - 8x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -8 \\ 1 \end{pmatrix}.$$

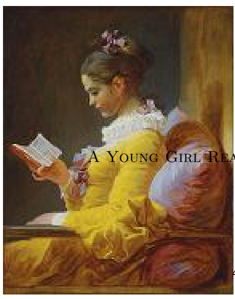
is the general solution in parametric form. It is easy to verify that $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ is indeed a concrete solution of this equation.

In summary, if there are solutions to linear system $A\vec{x} = \vec{b}$ and there is solution to the corresponding homogeneous system $A\vec{x} = \vec{0}$, then the solution to $A\vec{x} = \vec{b}$ can be either:

1. unique, when the homogeneous system only has trivial solution.
2. infinitely many, when homogeneous system only has non-trivial solution.

Reference

David C. Lay. Linear Algebra and Its Applications (3rd edition). Pages 50∼56.



A YOUNG GIRL READING, *by Fragonard*